MATH 2400: PRACTICE PROBLEMS FOR EXAM 1

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1) Find all real numbers x such that $x^3 = x$. Prove your answer!

Solution: If $x^3 = x$, then $0 = x^3 - x = x(x+1)(x-1)$. Earlier we showed using the field axioms that if if a product of numbers is equal to zero, then at least one of the factors must be zero. So we have $x = 0$, $x + 1 = 0$ or $x - 1 = 0$, i.e., $x = 0, \pm 1$. Conversely, these three numbers do indeed satisfy $x^3 = x$.

2) a) Prove that $\sqrt{6}$ is an irrational number. You may use the fact that if an integer x^2 is divisible by 6, then also x is divisible by 6. (For "extra credit", prove the fact of the previous sentence using the uniqueness of prime factorizations.)

Solution: let N be a positive integer. We claim that if N^2 is divisible by 6, then so is N. Indeed by the uniqueness of prime factorization, since N^2 is divisible by 2 and 3, 2 and 3 must each appear in the prime factorization of N.

Now we show that $\sqrt{6}$ is irrational. Seeking a contradiction, we suppose there are integers a, b, $b \neq 0$, such that $\sqrt{6} = \frac{a}{b}$. We may assume that a and b have no common prime divisor. Squaring both sides and clearing denominators gives $6b^2 = a^2$, so a^2 is divisible by 6. By the above paragraph this means a is divisible by 6: say $a = 6A$ for some integer A. Thus $6b^2 = a^2 = (6A)^2 = 36A^2$, so $6A^2 = b^2$. Thus b^2 is divisible by 6 and b is divisible by 6, contradicting the fact that a and b are relatively prime.

Comment: It is not in fact necessary to establish the fact that N^2 divisible by 6 implies N divisible by 6. Indeed, we can use the corresponding fact for divisibility by 2: if $6b^2 = a^2$ then a^2 is even, so a is even, so $a = 2A$ for some $A \in \mathbb{Z}$. Thus

$$6b^2 = a^2 = (2A)^2 = 4A^2,$$

and so

$$3b^2 = 2A^2.$$

So $3b^2$ is even. Since 3 is odd and the product of two odd numbers is odd, b^2 must be even and thus b is even. So a and b are both even, contradicting our assumption that they are relatively prime.

b) Show that if x^2 is an irrational number, so is x.

First Solution: Seeking a contradiction we suppose that x is rational, so $x = \frac{a}{b}$. Then $x^2 = \frac{a}{b}\frac{a}{b} = \frac{a^2}{b^2}$ would be rational, contradiction.

Second Solution: Rather than presenting the above argument as a proof by contradiction, we may also phrase it as a proof by **contrapositive**. Namely, the statement

$A \implies B$ is logically equivalent to its contrapositive statement not $A \implies$ not B. In this case, the contrapositive of the statement to be proved is: if x is rational, so is x^2. Notice that this is exactly what we showed above.

c) Show that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: Let $x = \sqrt{2}+\sqrt{3}$. By part b), it suffices to show that x^2 is irrational. Now $x^2 = (\sqrt{2}+\sqrt{3})^2 = 2+2\sqrt{6}+3 = 5+2\sqrt{6}$. If $5+2\sqrt{6} = \frac{a}{b}$, then $2\sqrt{6} = \frac{a}{b} - 5 = \frac{a-5b}{b}$ and thus

$$\sqrt{6} = \frac{a-5b}{2b}$$

would be a rational number, contradicting part a).

3) A subset S of the real numbers is **dense** if for any real numbers $\alpha < \beta$, there exists $x \in S$ such that $\alpha < x < \beta$.
a) Show that the set of rational numbers is dense. (Suggestion: make use of the fact that every real number has a decimal expansion.)

Solution: In class I sketched an argument using the existence of decimal expansions of real numbers. Here I will give a complete proof using the Archimedean Property of real numbers.
Step 1: It is no loss of generality to assume that $\alpha > 0$. Indeed, by the Archimedean property, for any $\alpha \in \mathbb{R}$ there exists an integer n such that $\alpha + n > 0$, and if x is a rational number such that $\alpha + n < x < \beta + n$, then $\alpha < x - n < \beta$, so $x - n$ is a rational number between α and β.
Step 2: Now suppose $\alpha > 0$. By the Archimedean property there exist positive integers n_1 and n_2 such that $\frac{1}{\alpha} < n_1$ and $\frac{1}{\beta-\alpha} < n_2$. Equivalently,

$$0 < \frac{1}{n_1} < \alpha$$

and

$$0 < \frac{1}{n_2} < \beta - \alpha,$$

so

$$0 < \frac{1}{n_1} + \frac{1}{n_2} < \beta.$$

Therefore the set

$$S = \{k \in \mathbb{Z}^+ \mid \frac{1}{n_1} + k\frac{1}{n_2} < \beta\}$$

is nonempty. By the Archimedean property S is finite, so has a largest element, say K. By construction $\frac{1}{n_1}+K\frac{1}{n_2} < \beta$. But moreover we must have $\alpha < \frac{1}{n_1}+K\frac{1}{n_2}$, for if not then $\frac{1}{n_1}+K\frac{1}{n_2} \leq \alpha$ and then $\frac{1}{n_1}+(K+1)\frac{1}{n_2} = (\frac{1}{n_1}+K\frac{1}{n_2})+\frac{1}{n_2} < \alpha+(\beta-\alpha) = \beta$, contradicting the definition of K. Thus

$$\alpha < \frac{1}{n_1} + K\frac{1}{n_2} < \beta.$$

Remark: Using the Archimedean Property to prove the density of $\mathbb{Q}$ in $\mathbb{R}$ is the best possible argument, in the following sense: the density of $\mathbb{Q}$ *does not* follow from the basic axioms (P0) through (P12) for ordered fields. In fact, in any ordered field which *does not* satisfy the Archimedean axiom, there exist **infinitesimal** elements

β, i.e., elements which are positive but less than $\frac{1}{n}$ for all $n \in \mathbb{Z}^+$. For such an infinitesimal element, the interval (β^2, β) does not contain any rational numbers! Of course it is a basic property of the real numbers that such infinitesimal elements do not exist!

b) Suppose $a, b \in \mathbb{Q}$ with $b \neq 0$. Show that $a + b\sqrt{2}$ is irrational.

Solution: This is essentially the same argument that we used in 2c) above. Let us write $a = \frac{x_a}{y_a}$, $b = \frac{x_b}{y_b}$, and suppose that $a + b\sqrt{2} = \frac{x_a}{y_a} + \frac{x_b}{y_b}\sqrt{2} = \frac{c}{d}$ with $c, d \in \mathbb{Z}$, $d \neq 0$. Then

$$\frac{x_b}{y_b}\sqrt{2} = \frac{c}{d} - \frac{x_a}{y_a} = \frac{cy_a - dx_a}{dy_a},$$

so

$$\sqrt{2} = \frac{y_b(cy_a - d)}{dx_b y_a}$$

would be rational, contradiction.

c) Show that the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is dense.

Solution: Let $\alpha < \beta$ be real numbers. By part a), there exists a rational number a such that $\alpha < a < \beta$: fix such an a. Similarly, by the Archimedean property of real numbers, there exists a positive integer n such that

$$\frac{\sqrt{2}}{\beta - a} < n;$$

thus

$$0 < \frac{1}{n}\sqrt{2} < \beta - a,$$

and finally

$$\alpha < a < a + \frac{1}{n}\sqrt{2} < \beta.$$

d) Conclude that the set of irrational numbers is dense.

Solution: Note that we proved something slightly stronger than was required in part c): the set of numbers of the form $a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$ and $b \neq 0$ is dense in $\mathbb{R}$. By part b), all numbers of this form are irrational. So every interval (α, β) with $\alpha < \beta$ contains an irrational number: done.

4) Prove that for any real numbers x, y, $||x| - |y|| \leq |x - y|$.

Solution: This is proven (twice) on p.11 of `www.math.uga.edu/~pete/math2400_lecture_1.pdf`.

5) State the Principle of Mathematical Induction and the Principle of Strong/Complete Induction.

Solution: Let $P(n)$ be a statement defined for each positive integer n.

Principle of Induction: Suppose that:
a) $P(1)$ is true, and
b) For all $n \in \mathbb{Z}^+$, $P(n) \implies P(n+1)$.

Then $P(n)$ holds for all $n \in \mathbb{Z}^+$.

Principle of Strong/Complete Induction: Suppose that:
a) $P(1)$ is true, and
b) For all $n \in \mathbb{Z}^+$, if for all $1 \leq k \leq n$ $P(k)$ holds, then $P(n+1)$ holds.

6) Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be n different functions, and let $c \in \mathbb{R}$. Suppose that for all $1 \leq i \leq n$, $\lim_{x\to c} f_i(x) = L_i$. Show that $\lim_{x\to c} f_1(x)f_2(x)\cdots f_n(x) = L_1L_2\cdots L_n$. (Suggestion: use the product rule for limits and induction on n.)

Solution: We want to show this for all integers $n \geq 2$, so we go by induction on n.
Base Case ($n = 2$): This is the product rule for limits. If $\lim_{x\to c} f_1(x) = L_1$ and $\lim_{x\to c} f_2(x) = L_2$, then $\lim_{x\to c} f_1(x)f_2(x) = L_1L_2$.
Induction Step: Suppose that for any $n \geq 2$, then the limit of the product of any n functions in the product of the limits of the individual functions. Now let $f_1(x), \ldots, f_{n+1}(x)$ be functions such that for all $1 \leq k \leq n+1$, $\lim_{x\to c} f_i(x) = L_i$. Then

$$\lim_{x\to c} f_1(x)\cdots f_n(x)f_{n+1}(x) = \lim_{x\to c}(f_1(x)\cdots f_n(x))f_{n+1}(x)$$
$$= \lim_{x\to c}(f_1(x)\cdots f_n(x)) \lim_{x\to c} f_{n+1}(x) = (L_1\cdots L_n)\cdot L_{n+1} = L_1\cdots L_n\cdot L_{n+1}.$$

Above we applied first the case $n = 2$ (i.e., the product rule for limits) and then the induction hypothesis.

7) We define a sequence of numbers $x_1, x_2, \ldots, x_n$ recursively, as follows: $x_1 = 0$, and for all $n \geq 1$, $x_{n+1} = 2x_n + 1$.
a) Compute the first 8 terms of the sequence.

Solution: We have

$$x_1 = 0,\ x_2 = 1,\ x_3 = 3,\ x_4 = 7,\ x_5 = 15,\ x_6 = 31,\ x_7 = 63,\ x_8 = 127.$$

b) Find a closed form expression for x_n and prove it by induction.

Solution: Based on the above data we guess that $x_n = 2^{n-1} - 1$ for all $n \in \mathbb{Z}$.
Base Case ($n = 1$): $0 = x_1 = 2^{1-1} - 0$.
Induction Step: Let $n \in \mathbb{Z}^+$, and suppose $x_n = 2^{n-1} - 1$. Then

$$x_{n+1} = 2x_n + 1 = 2\left(2^{n-1} - 1\right) + 1 = 2^n - 2 + 1 = 2^{(n+1)-1} - 1.$$

By the Principle of Mathematical Induction we have $x_n = 2^{n-1} - 1$ for all n.

8) Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
a) Show that if f is odd, $f(0) = 0$.

Solution: We have $f(0) = f(-0) = -f(0)$, so $2f(0) = 0$ and thus $f(0) = 0$.

b) Suppose that f is even and f is differentiable. Give a geometric explanation of why $f'(0) = 0$. (Suggestion: say something about slopes of secant lines.)

Solution: Let δ be a small positive number. Then the secant line from $(-\delta, f(-\delta))$

to $(0, f(0))$ is is the reflection through the y-axis of the secant line from $(0, f(0))$ to $(\delta, f(\delta))$, so the slope of the first secant line is equal to the negative of the slope of the second secant line. But in order for the function to have a well-defined tangent line at zero, these two slopes must approach the same value as $\delta \to 0$. The only slope m such that $m = -m$ is $m = 0$.

c) Give an example of an even function which is *not* differentiable at $x = 0$.

Solution: $f(x) = |x|$.

9) a) Sketch the graphs of $y = |\sin x|$, $y = \sin^2 x$ and $y = \sin(\frac{1}{x})$.

Solution: Omitted. You can easily find an online graphing utility on the internet and check your answers here.

b) Suppose you are given the graph of $y = f(x)$. Explain how to obtain the graph of $y = |f(x)|$.

Solution: Each portion of the graph of $y = f(x)$ which lies above the x-axis remains unchanged, whereas each portion of the graph of $y = f(x)$ which lies below the x-axis gets rotated about the axis 180 degrees, or equivalently reflected through the axis.

c) Use part b) to give a geometric explanation (no ϵ's and δ's required!) of why $y = f(x)$ continuous implies $y = |f(x)|$ continuous.

Solution: If the graph of $y = f(x)$ is an unbroken curve and we "twist" certain portions of it by reflecting them through the x-axis yields an unbroken curve.

d) If $y = f(x)$ is differentiable, must $y = |f(x)|$ be differentiable? If not, explain how to find the points of non-differentiability of $y = |f(x)|$. (Again, formal proofs are not required here.)

Solution: If $y = f(x)$ is differentiable, then $y = |f(x)|$ is differentiable at any point a such that $f(a) \neq 0$. However, if $f(a) = 0$, then $y = |f(x)|$ is differentiable at a if and only if $f'(a) = 0$: otherwise the twisting process creates a "cusp" at a where the slopes of the tangent lines from the left and the right have opposite signs.

10) a) Give the ϵ-δ definition of "$\lim_{x\to c} f(x) = L$".

Solution: For all $\epsilon > 0$, there exists $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, $|f(x) - L| < \epsilon$.

b) Give the ϵ-δ definition of "f is continuous at $x = c$".

Solution: For all $\epsilon > 0$, there exists $\delta > 0$ such that for all x with $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$.

c) Using the definitions of parts a) and b), prove that f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

Solution: First suppose that f is continuous at c according to the above ϵ-δ definition. It is then immediate that by taking $L = f(c)$ we have $\lim_{x \to c} f(x) = f(c)$. Conversely, suppose that $\lim_{x \to c} f(x) = f(c)$: then, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. To square this with the above ϵ-δ definition of continuity we need only observe that when $x = c$ we have $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$.

11) Show that the following limits exist directly from the ϵ-δ definition.
a) $\lim_{x \to 4} 3x - 19 = -7$.

Solution: For any $\epsilon > 0$, take $\delta = \frac{\epsilon}{3}$. Then: if $0 < |x - 4| < \delta$,

$$|3x - 19 - (-7)| = |3x - 12| = 3|x - 4| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

b) $\lim_{x \to 2} x^3 = 8$.

Solution: For any $\epsilon > 0$, take $\epsilon = \min(1, \frac{\delta}{19})$. If $0 < |x - 2| < \delta$, then in particular $|x - 2| \leq 1$, so $|x| \leq 3$, and thus

$$|x^3 - 8| = |x - 2||x^2 + 2x + 4| < (|x|^2 + 2|x| + 4)\delta$$
$$\leq (3^2 + 2 \cdot 3 + 4)\delta = 19\delta \leq 19 \cdot \frac{\epsilon}{19} = \epsilon.$$